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New exactly solvable orbifold models

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Abstract. We employ an 'orbifold procedure' to construct new exactly solvable lattice statistical mechanical models. Starting with the $\widehat{\mathrm{su}(3)}$ models at level $k = 3j$, with j an integer, we obtain new models with the same bulk free energy. The bulk free energy is known for the $\widehat{\mathrm{su}(n)}$ models in a two-parameter subspace of the full parameter space, so the new models are also exactly solvable in this subspace. We express the toroidal partition function of the original model as a sum over partition functions of the orbifold model with twisted boundary conditions. Each model has a critical point, described by a conformal field theory with central charge $c = 2[1 - 12/(k+3)(k+2)]$.

1. The orbifold incidence diagrams

A method for constructing new two-dimensional statistical mechanical lattice models, called the 'orbifold' procedure, was defined by Fendley and Ginsparg (1989), hereafter referred to as FG. Motivated by the construction of orbifolds in conformal field theory (Dixon *et al* 1986), it involves taking a known model and 'modding out' by a symmetry, resulting in the orbifold, a new model with the same bulk free energy. Thus if the original model is exactly solvable, then the orbifold is as well. In the simplest cases, e.g. the Ising model, the orbifold is the Kramers–Wannier dual of the original. This technique was used to relate a number of known models to one another, including the A_N series (Andrews *et al* 1984) to the D_N series (Pasquier 1987) and the eight-vertex model (Baxter 1982) to the $\hat{D}_N$ series (Akutsu *et al* 1986). In this paper we will construct new exactly solvable models using the orbifold technique, starting with the models solved by Jimbo *et al* (1987).

These models are associated with the affine algebras $\widehat{\mathrm{su}(n)}$ at level k. The models for $\widehat{\mathrm{su}(2)}$ are the A_N series, where $N = k + 1$. We will discuss only the $\widehat{\mathrm{su}(3)}$ case, but these arguments generalise to arbitrary n. We will show that orbifolding the $\widehat{\mathrm{su}(3)}$ model when k is a multiple of 3 results in a new exactly solvable model. This is completely analogous to the construction of a D_N model from A_{2N-3}, indicating that the ability to construct seemingly different but deeply related models is a general property of lattice models with particular types of symmetry. This was shown to be possible at criticality for all n when the level is a multiple of n by Kostov (1988).

All the above lattice models can be represented as 'face' models. They are defined by placing a 'height' on each site of a square lattice. We work on a lattice rotated by 45 degrees. Each model has an associated 'incidence diagram', restricting the allowed heights. In the $\widehat{\mathrm{su}(3)}$ models, the allowed heights are labelled by the representations of this affine algebra at level k. The incidence diagram for level 3 is displayed in figure 1;

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the general diagram for $\widehat{su(3)}$ level k has $k+1$ nodes on each side of the triangle. Each node corresponds to an allowed height in the model. Allowed height configurations are those for which heights on nearest-neighbour sites correspond to nodes that are connected on the diagram; in addition, we draw an arrow on each link of the lattice corresponding to the arrows between the nodes of figure 1, and these arrows must point to the upper left or upper right. One may think of the heights or of the links as the dynamical variables. A given height lies on one of three sublattices: for $k = 3$ only heights 1_r occur on lattice 1, only 2_r on lattice 2, and only $(0_r, 3)$ on lattice 3.

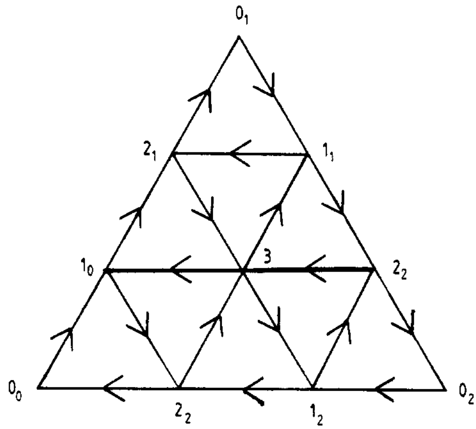

Figure 1. The $\widehat{su(3)}$ level 3 incidence diagram.

We assign a Boltzmann 'weight' $W_{ac}^{(db)}$ to each face of the lattice, where a, b, c and d are the heights clockwise around the face starting from the lowermost corner. The partition function Z is

$$Z = \sum_{\{\text{configs}\}} \prod_{\text{faces}} W_{ac}^{(db)} \tag{1}$$

where the sum is over all allowed height configurations (alternatively, allowed link configurations), and the product is over all faces. If the weights are a function of a parameter u and obey the star–triangle relations (Yang–Baxter equation), then the model can be solved exactly (Baxter 1982). Jimbo *et al* (1987) found a two-parameter set of weights for the $\widehat{su(n)}$ models that obey the star–triangle relations.

The weights for the $\widehat{su(3)}$ models have a $\mathbf{Z}_3$ symmetry, which is generated by rotating the incidence diagram by 120 degrees. For the $k = 3$ case displayed in figure 1, this means that changing $h_r \to h_{r+1}$ leaves the weights invariant, where $h = 0, 1, 2$, and r is defined mod 3. The height 3 is a fixed point of this symmetry.

We will orbifold the models for which k is a multiple of 3 by this $\mathbf{Z}_3$ symmetry. In this case, as in all models suitable for orbifolding, the symmetry only relates heights on the same sublattice. As explained in FG, orbifolding requires first 'modding out' by a symmetry to obtain a new incidence diagram, and then defining the weights for the orbifold by taking particular linear combinations of those for the original model. Finding the new incidence diagram is a simple graphical procedure. Heights that are related by the symmetry in the original model are identified, while heights that are left fixed are replicated so that they form an orbit of the group in the new model. When $k = 3$, this means that the new model has heights $\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}_0, \tilde{3}_1$ and $\tilde{3}_2$, where a new $\mathbf{Z}_3$ cyclically permutes the $\tilde{3}_r$. (We denote orbifold heights with a tilde.) Constructing the new incidence diagram, however, involves subtleties not present in the models discussed in FG. These incidence diagrams were first discussed by Kostov (1988).

In order to find the incidence diagram, we look at a subdiagram that we call the 'prototypical' model. It has the incidence diagram displayed in the left half of figure 2. We orbifold the prototypical model using the procedure discussed in the preceding paragraph. This identifies heights $h_r \to \tilde{h}$ while replicating height 3 to $(\tilde{3}_0, \tilde{3}_1, \tilde{3}_2)$; the result is displayed in the right half of figure 2.

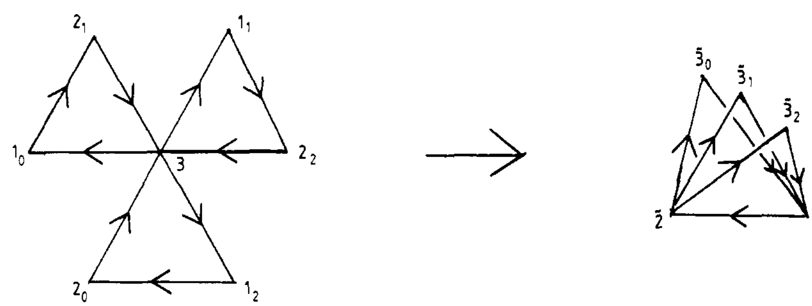

Figure 2. The orbifolding of the incidence diagram for the 'prototypical' model for the $\widehat{su(3)}$ models.

It is useful here to think of the dynamical variables as being the links, not as the heights. To obtain the orbifold of the $k = 3$ model, we include the rest of the links, which we call 'spectator' links. These are the nine links connecting heights $(0_r, 1_r, 2_r)$. The identification of heights leaves only three spectator links, connecting heights $(\tilde{0}, \tilde{1}, \tilde{2})$. We then obtain the incidence diagram shown in figure 3.

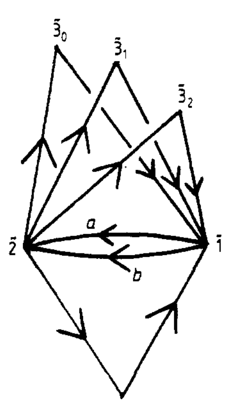

Figure 3. The incidence diagram for the orbifold of the $k = 3$ model.

Notice that there are two different links connecting the heights 1 and 2, because in the original model there are two links connecting heights 1_r and 2_s that are unrelated by the $\mathbf{Z}_3$ symmetry. The dynamical variables in the orbifold are the links and not the heights: the weights differ according to whether link a or link b is placed on the lattice, and the partition sum must be taken over both. Models of this kind are discussed by Pasquier (1988).

We can now find the incidence diagrams for all the $\mathbf{Z}_3$ orbifolds of models with $k = 3j$. To specify the spectator links, we draw three diamonds starting at each corner of the incidence diagram and ending at the centre height, which we call f. Each side of the diamond has j links and $j + 1$ nodes. The spectator links then are all the links on and inside the diamonds, with the exception of the six links that touch f (two from each diamond). This situation for the $k = 6$ case is illustrated in figure 4.

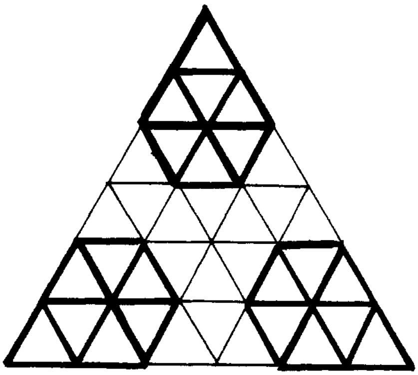

Figure 4. The incidence diagram for the $k = 6$ model. We omit arrows for convenience. The spectator links are the thick lines.

The links that remain are an extension of the prototypical diagram in figure 2. We orbifold these as above: the centre height f is the only height fixed under the $\mathbf{Z}_3$, so all the heights except for f are identified with the heights in the same $\mathbf{Z}_3$ orbit, and f is replicated to $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2$. To combine this with the spectator links, we merely identify the heights related by the $\mathbf{Z}_3$ symmetry as before, remembering not to identify links unrelated by the symmetry. Thus the incidence diagram resembles a cone with a flap sticking out of it, with the heights $\tilde{f}_r$ at the point, the spectator links on the side of the cone with the flap, and the remaining links on the other side. We display the $k = 6$ case in figure 5.

2. Constructing orbifold weights using a graphical decomposition

The most general weights for the $\widehat{\mathrm{su}(3)}$ level 3 model with $\mathbf{Z}_3$ symmetry are

$$W^{(32_r)} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \qquad W^{(1_r3)} = \begin{pmatrix} J & K \\ L & M \end{pmatrix} \tag{2a}$$

$$W_{33}^{(2_r1_{r+1})} = P \qquad W_{33}^{(2_r1_{r+2})} = Q$$

$$W^{(2_r1_r)} = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \tag{2b}$$

$$W_{1_r1_r}^{(0_r2_{r+1})} = A \qquad W_{1_r1_r}^{(0_r2_r)} = B$$

$$W_{2_r2_r}^{(1_{r+2}0_r)} = C \qquad W_{2_{r+2}2_{r+2}}^{(1_r0_r)} = D \tag{2c}$$

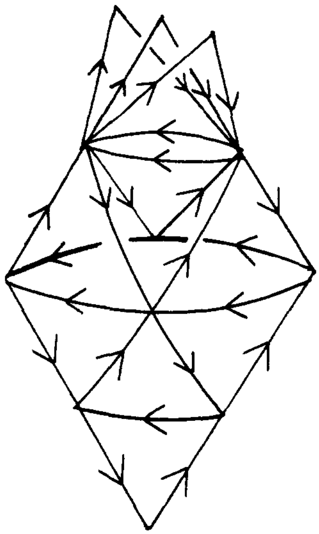

Figure 5. The incidence diagram for the orbifold of the $k = 6$ model.

where the rows and columns of $W^{(32_r)}$ are indexed by heights 1_{r+2} and 1_r, $W^{(1_r3)}$ by 2_{r+1} and 2_r, and $W^{(2_r1_r)}$ by 0_r and 3. Jimbo *et al* (1987) have solved this model in a two-parameter subspace. Their values for the weights (using the parametrisation of Jimbo *et al* (1988)) are

$$
\begin{aligned}
&A = C = P = \frac{[1+u]}{[1]} && B = D = Q = \frac{[1-u]}{[1]} \\
&E = J = \frac{[2-u]}{[2]} && H = M = \frac{[2+u]}{[2]} \\
&F = G = K = L = \frac{[u][3]^{1/2}}{[2][1]^{1/2}} && \\
&R = U = \frac{[3-u]}{[3]} && S = T = \frac{[u][2]}{[3][1]}
\end{aligned}
\qquad (3)
$$

where $[u] = \theta_1(\pi u/6, q)$ and

$$\theta_1(v,q) = 2|q|^{1/8}\sin v \prod_{n=1}^{\infty}(1 - 2q^n\cos 2u + q^{2n})(1 - q^n)$$

is an elliptic theta function. The parameter q describes the temperature; $q \to 0$ is the critical limit. The parameter u describes the anisotropy and does not affect the critical properties.

We use a graphical decomposition of the partition function of the original model in order to express the orbifold weights in terms of the original weights, and to relate their partition functions. We define a $\mathbf{Z}_3$ charge $\chi(h_r) \equiv \zeta^r$ at each lattice site with height h_r, where $\zeta = e^{2\pi i/3}$. Configurations of the original model are broken into clusters of distinct charge. We represent each configuration graphically by drawing double lines on the links of the lattice on which spectator links have been placed; sites connected by double lines have the same charge. The partition function then is rewritten as a triple sum

$$\sum_{\text{graphs}} \sum_{\substack{\text{configs}\\ \text{internal}\\ \text{to clusters}}} \sum_{\substack{\text{cluster}\\ \text{charges}}} \prod_{\text{faces}} W_{ac}^{(db)}. \tag{4}$$

The third sum is not an unrestricted sum—a cluster's charge will depend on charges of neighbouring clusters.

We start by constructing the weights and relating the toroidal partition functions of the prototypical model and its orbifold, as displayed in figure 2. The only weights that occur here are E, J, P, Q and U. The first two sums in (4) are trivial, because there is only one graph: no double lines occur because there are no spectator links. We rewrite the partition sum using

$$W_{33}^{(2_r 1_s)} = \tfrac{1}{3}(U + P + Q) + \chi(2_r)\chi^*(1_s)\tfrac{1}{3}(U + \zeta P + \zeta^* Q) + \chi^*(2_r)\chi(1_s)\tfrac{1}{3}(U + \zeta^* P + \zeta Q). \tag{5}$$

We substitute (5) into (4) and expand out the product into a sum of terms, each of which is a product with a single contribution from each face. We represent these terms graphically by drawing a single vertical line with an upward arrow on a face when the second of the three terms in (5) contributes for that face, a vertical line with a downward arrow when the third contributes, and no line when the first contributes. A typical graph is displayed in figure 6, where we have indicated the presence of the three sublattices discussed above. To obtain the full partition function, we must sum over all graphs (i.e. all possible placements of arrows) as well as over the cluster charges. For a fixed graph we perform the sum over charges. This is easy, because the heights on each horizontal row of links between lattices 1 and lattice 2 must have related $\mathbf{Z}_3$ charge. We call these rows the 1–2 rows, and they are denoted by dotted lines in figure 6. Specifying the charge at a single site on a 1–2 row fixes the charges on the rest of the row, so the charge sum in (4) is performed over each 1–2 row independently. Since $1 + \zeta + \zeta^* = 0$, this sum is zero unless the number of arrows pointing into each 1–2 row is equal to the number pointing out. In this case, the charge sum merely gives a factor of three for each 1–2 row.

After summing over cluster charges, we identify each graph with a *configuration* in the orbifold. The sum over graphs then becomes the configuration sum. All the sites on lattice 1(2) have height $\tilde{1}(\tilde{2})$. The single lines are domain walls separating regions of heights $\tilde{3}_r$ on lattice 3: going to the right across an upward-pointing arrow changes $\tilde{3}_r$ to $\tilde{3}_{r+1}$, while going to the right across a downward-pointing arrow changes $\tilde{3}_r$ to $\tilde{3}_{r-1}$. Domain walls only affect the heights $\tilde{3}_r$ on that horizontal row. This assignment of orbifold heights to a particular graph can always be done locally, but in the orbifold we have a vertical seam across which height $\tilde{3}_r$ is matched with $\tilde{3}_{r+t}$, where t is the number of upward arrows minus the number of downward arrows in the row. The restriction on allowed graphs discussed above forces t to be the same for all rows. We refer to such a configuration as being twisted by t. Also notice that given a graph, there are three different allowed orbifold configurations for each row, corresponding to shifting every height on the row from $\tilde{3}_r$ to $\tilde{3}_{r+s}$, where $s = 0, 1, 2$. The orbifold weights

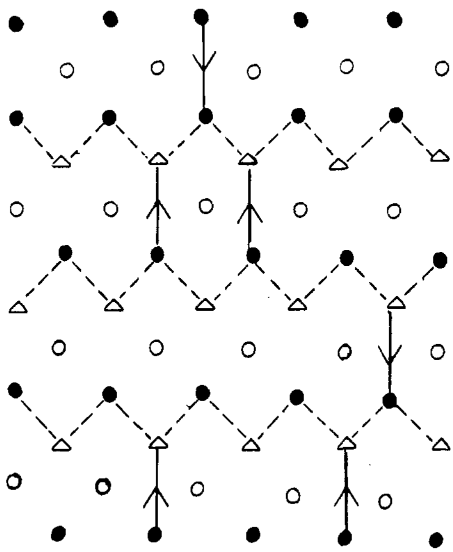

Figure 6. A non-zero graph in the decomposition of the prototypical model. Lattice 3 sites are denoted with unfilled circles, lattice 2 sites with solid circles and lattice 1 sites with triangles. The 1–2 rows are denoted by dotted lines.

are easily determined. Faces with links in a 1–2 row are unaffected by the graphical expansion of $W_{33}^{(2_r 1_s)}$, so

$$\widetilde{W}_{\widetilde{11}}^{(\widetilde{3_r}\widetilde{2})} = E \qquad \widetilde{W}_{\widetilde{22}}^{(\widetilde{13_r})} = J. \tag{6a}$$

Each single line has a contribution corresponding to the appropriate term in (5). Thus the weights on these faces are

$$\widetilde{W}^{(\widetilde{21})} = \frac{1}{3}\begin{pmatrix} U+P+Q & U+\zeta P+\zeta^* Q & U+\zeta^* P+\zeta Q \\ U+\zeta^* P+\zeta Q & U+P+Q & U+\zeta P+\zeta^* Q \\ U+\zeta P+\zeta^* Q & U+\zeta^* P+\zeta Q & U+P+Q \end{pmatrix} \tag{6b}$$

where the rows and columns are indexed by the heights $\tilde{3}_r$.

We have shown that each graph in the expansion of the original model corresponds to a (possibly twisted) height configuration in the orbifold. This correspondence is not one-to-one—as discussed above, there are three possible assignments of heights $\tilde{3}_r$ on each row. Thus the correspondence is one-to-3^m, where m is the number of lattice 3 rows. We have chosen the weights in (6a, b) so that the contribution to the partition function from each graph in the original model is 3^m times the contribution from one corresponding orbifold configuration, the 3^m arising from the factor of three from the charge sum on each 1–2 row. Thus the contribution from each graph is equal to the contribution of all corresponding orbifold height configurations, and

$$Z = \sum_{t=0,1,2} \tilde{Z}(t) \tag{7}$$

where $\tilde{Z}(t)$ is the partition function of the orbifold model with horizontal twist t. This equation also holds when the models are defined on a cylinder. To write this in a more familiar form, we notice that if we twist in the vertical direction, the partition function does not change (it only affects weights on faces of type 1 and 2, and these are always E and J). Thus

$$Z = \frac{1}{3} \sum_{t_h, t_v = 0,1,2} \widetilde{Z}(t_h, t_v). \tag{8}$$

Since the bulk free energy ($\equiv \lim_{N\to\infty} N^{-1} \ln Z$, where N is the number of sites) should not depend on the boundary conditions, (8) shows that the bulk free energies of the two models are equal.

3. The orbifold weights

In this section we prove that relation (8) holds for $\widehat{\mathrm{su}(3)}$ at level $k = 3j$ and its orbifold. The weights of the orbifold in the $k = 3$ case are

$$\widetilde{W}^{(\widetilde{3_r 2})}_{\widetilde{11}} = \begin{pmatrix} E & \zeta^r F \\ \zeta^{-r} G & H \end{pmatrix} \qquad \widetilde{W}^{(\widetilde{13_r})}_{\widetilde{22}} = \begin{pmatrix} J & \zeta^{-r} K \\ \zeta^r L & M \end{pmatrix} \tag{9a}$$

$$\widetilde{W}^{(\widetilde{21})} = \frac{1}{3} \begin{pmatrix} 3R & \sqrt{3}S & \sqrt{3}S & \sqrt{3}S \\ \sqrt{3}T & U+P+Q & U+\zeta P+\zeta^* Q & U+\zeta^* P+\zeta Q \\ \sqrt{3}T & U+\zeta^* P+\zeta Q & U+P+Q & U+\zeta P+\zeta^* Q \\ \sqrt{3}T & U+\zeta P+\zeta^* Q & U+\zeta^* P+\zeta Q & U+P+Q \end{pmatrix} \tag{9b}$$

$$\widetilde{W}^{(\widetilde{02})}_{\widetilde{11}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad \widetilde{W}^{(\widetilde{10})}_{\widetilde{22}} = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \tag{9c}$$

where the matrix in (9b) is indexed by the heights $\widetilde{0}, \widetilde{3}_r$. The other matrices are indexed by by the links a and b; each of these faces has two links between heights $\widetilde{1}$ and $\widetilde{2}$, so there are four possible configurations on the face.

We use the above graphical representation of placing double lines on the spectator links, and expand out Z using (5). The first and third sums in (4) are relevant here. Again, we identify an orbifold height configuration with every graph. The effect of including the spectator link connecting 1_r and 2_r is easy to handle. Every time this link appears, we assign a link b in the orbifold configuration; every time the link connecting 1_r and 2_{r+1} appears we assign a link a in the orbifold. This determines the weights (9a). The powers of ζ in (9a) (which do not appear in any of the models discussed in FG) insure that this assignment is globally one-to-one. To see this, look at a graph in the original model without any heights 0_r. Every time link a appears, the charge on that 1–2 row changes. Thus only certain combinations of links a and b on each 1–2 row are allowed, insuring that each site has a well defined charge. This restriction is implemented in the orbifold by the sum over the three allowed assignments of heights $\widetilde{3}_r$ on the lattice 3 row above each 1–2 row. For allowed configurations, the powers of ζ in (9a) cancel and we get the factor of three discussed above. We must sum over the assignments on the top and bottom rows of the lattice independently, so (7) holds only for cylindrical boundary conditions, not for toroidal.

Orbifold weights involving the heights 0_r are in one-to-one correspondence with the original weights involving them, once we take into account the presence of links a and b. This results in weights (9c) and in those in the first row and the first column of

(9b). The same factors of ζ discussed in the preceding paragraph and the off-diagonal zeros in (9c) insure that this is a one-to-one assignment. The remaining weights in (9b) result from the same argument as in the prototypical model.

To relate the partition functions, we need to account for factors of three beyond those dealt with in the prototypical model. Fix a term in the first and second sums in (4). As before, specifying the charge at a single site on a 1–2 row determines the charges on the rest of the row. The weights are independent of this specification, so when no heights 0_r are present, the factors of three work out as in the prototypical model. When a height 0_r is present between two 1–2 rows, the charges on these two rows are no longer independent. This removes a factor of three, which is compensated in the orbifold by the two powers of $1/\sqrt{3}$ that appear, using weights (9b). If we add a height 0_s to a row already with a 0_r in it, then no new factors of 1/3 appear in the original model. In the new model, we get an additional $(1/\sqrt{3})^2$, which is cancelled by the new factor of three resulting from the three different ways of assigning heights 3_r on the sites in between the two heights 0_r and 0_s. The horizontal twist sectors must be included as before.†

We have therefore shown that (8) holds for the $k = 3$ model and its orbifold. The bulk free energies of the two models are therefore equal, so the energy critical exponents must be the same. Since the toroidal partition functions are not equal, their operator contents must differ. When the original weights obey (3), the orbifold is exactly solvable as well. Notice that even though powers of $\zeta = e^{2\pi i/3}$ appear in the weights, the models still have positive energy: powers of ζ in (9a) cancel in allowed configurations, so reversing all vertical arrows gives the configuration with complex conjugate weight. Including both configurations results in a real contribution to the partition function.

To generalise the weight assignments to arbitrary k is simple. All three sums in (4) are relevant. To find the weights, we fix a term in the first two sums, and make the one-to-one assignments as before. Weights involving the fixed height f are exactly as in (9b). An orbifold weight for a face consisting entirely of what were originally spectator links is equal to the original weight; the same goes for a weight made entirely of what were originally non-spectator links. These are analogous to (9c) and the diagonal terms in (9a). Weights with both kinds of links are multiplied by powers of ζ like the off-diagonal terms in (9a). The proof of (8) is identical to the one just given.

We note in closing that the continuum limits of the critical points of the $\widehat{\mathrm{su}(3)}$ models are given by the conformal field theories with an extended symmetry algebra discussed by Fateev and Zamolodchikov (1987). These theories have central charge $c = 2(1 - 12/(k + 2)(k + 3))$. At criticality, our orbifold procedure reduces to the conventional conformal field theory notion. Orbifolding does not change the central charge, so the new models have these values as well. The orbifold partition function at criticality has been explicitly calculated by Kostov (1988), and was shown to be a sum of Gaussian partition functions. Critical $\mathbf{Z}_3$ orbifold lattice models have also been considered by DiFrancesco and Zuber (1989).

† Rows with a height 0_r are allowed any number of vertical lines. To keep the factors of three correct, we must choose only one of the three possible twists for that row. We make these choices so that there is a consistent twist along the entire vertical seam.

Acknowledgments

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